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LETTER TO THE EDITOR

On the nature of the critical point in the three-spin triangular Ising model

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Abstract. The nature of the critical point in the triangular Ising model with pure triplet interactions is discussed. In particular, it is shown that the correction terms to the asymptotic power law behaviour of thermodynamic quantities require the existence of a third relevant scaling field in accord with recent renormalization group calculations. The exponent of this field is found to be $y_3 = 7/8$.

Two groups (den Nijs *et al* 1976, Imbro and Hemmer 1976) have recently applied renormalization group techniques to elucidate the critical behaviour of the Ising model on the triangular lattice with both pair and triplet interactions. Although these calculations are only approximate and, indeed, of relatively poor numerical accuracy, the qualitative behaviour revealed by each is very similar. In particular, both investigations found that the pure three-spin limit (Baxter and Wu 1973) corresponds to a fixed point with *three* relevant eigenvalues. In this letter we explore some of the consequences of this conclusion. Most significantly, we find that the corrections to the asymptotic behaviour of known or conjectured thermodynamic quantities can only be accounted for if the basic conclusion is valid. These correction terms can then be used to determine the value of the third eigenvalue explicitly.

To be more specific we consider an Ising model on a triangular lattice with reduced Hamiltonian

$$H = K_1 \sum \sigma_i + K_2 \sum \sigma_i\sigma_j + K_3 \sum \sigma_i\sigma_j\sigma_k, \tag{1}$$

where the first sum runs over all sites of the lattice, the second over all nearest-neighbour bonds and the third over all elementary triangles of the lattice. As usual, we absorb a factor of $-\beta$ $(= -1/k_{\mathrm{B}}T)$ into the definition of H. The system described by (1) is known to exhibit at least three critical points with distinct sets of critical exponents. At $K_1 = K_3 = 0$, $K_2 = K_{2,\mathrm{c}} = \frac{1}{4}\ln 3 \simeq 0{\cdot}274$, the specific heat diverges logarithmically (Wannier 1950, Houtappel 1950), while at $K_1 = K_2 = 0$, $K_3 = \pm K_{3,\mathrm{c}} = \pm\frac{1}{2}\ln(1+\sqrt{2}) \simeq \pm 0{\cdot}4407$, the specific heat exponent has the value $\alpha = 2/3$ (Baxter and Wu 1973, 1974). Presumably these two distinct types of critical behaviour correspond to different fixed points. One of the objectives of the renormalization group calculations referred to above was to investigate the domains of attraction or critical surfaces associated with these fixed points and to explore the critical behaviour of (1) in the full space (K_1, K_2, K_3).

The salient findings of den Nijs *et al* (1976) and Imbro and Hemmer (1976) can be summarized as follows. Renormalization group transformations of (1) do exhibit three non-trivial fixed points, which can be identified with the exactly known critical points. Following Imbro and Hemmer (1976) we shall refer to these as the Onsager fixed point (for which $K_2^* \neq 0$, $K_1^* = K_3^* = 0$) and the Baxter–Wu fixed points ($K_1^* = K_2^* = 0$, $K_3^* \neq 0$). The former has two relevant eigenvalues, while the Baxter–Wu points have three. Thus in the space (K_1, K_2, K_3), (1) exhibits a critical line passing through the Onsager fixed point and terminating at the two Baxter–Wu points. All points on this line, except for the two end points, map to the Onsager fixed point and thus exhibit conventional Ising-like critical behaviour.

In the vicinity of one of the Baxter–Wu fixed points, the singular part of the free energy can be written in the scaled form (see, e.g., den Nijs *et al* 1976)

$$-\beta f_s(K_1, K_2, K_3) \simeq |g_t|^{d/y_t} Q(u, v), \tag{2}$$

with

$$u = g_h/|g_t|^{\Delta}, \qquad v = g_3/|g_t|^{\Delta_3}, \tag{3}$$

where

$$\Delta = y_h/y_t, \qquad \Delta_3 = y_3/y_t. \tag{4}$$

In these expressions g_t, g_h, g_3 denote the three relevant non-linear scaling fields associated with the fixed point and y_t, y_h, y_3 their respective critical exponents, i.e. under a renormalization group transformation with spatial rescaling factor l, g_α transforms as $g_\alpha l^{y_\alpha}$. In terms of the basic coupling constants these fields can be expanded (Wegner 1972) as

$$\begin{aligned} g_t &= K_3 - K_{3,c} + \ldots, \\ g_h &= K_1 + aK_2 + \ldots, \\ g_3 &= K_2 + bK_1 + \ldots, \end{aligned} \tag{5}$$

where by symmetry the corrections are at least quadratic in K_1, K_2 and $\Delta K_3 = K_3 - K_{3,c}$ (see den Nijs *et al* 1976). Thus y_t and y_h are determined from standard critical exponents,

$$y_t = 1/(2-\alpha) = 3/2, \qquad y_h = 2 - y_t\beta = 15/8, \tag{6}$$

where we have used the result of Baxter and Wu (1973, 1974) for α and the very plausible conjecture of Baxter *et al* (1975) for β. The exponent y_3 does not influence the asymptotic critical behaviour. However, as we shall see, it does affect correction terms.

Explicitly we consider the critical behaviour of

$$M_0(K_3) = \lim_{K_1 \to 0+} \left(\frac{\partial}{\partial K_1} - \beta f_s(K_1, 0, K_3) \right), \tag{7}$$

and

$$P_0(K_3) = \lim_{K_2 \to 0+} \left(\frac{\partial}{\partial K_2} - \frac{1}{3} \beta f_s(0, K_2, K_3) \right), \tag{8}$$

both of which vanish identically for $K_3 \leqslant K_{3,c}$ (i.e. $T \geqslant T_{3,c}$) and are non-zero for

$K_3 > K_{3,c}$ ($T < T_{3,c}$). Differentiating (2) implies that

$$M_0 \simeq |g_t|^{\frac{4}{3}-\Delta} Q_u(0,0) + b|g_t|^{\frac{4}{3}-\Delta_3} Q_v(0,0) + \dots \tag{9}$$

$$P_0 \simeq \tfrac{1}{3} a |g_t|^{\frac{4}{3}-\Delta} Q_u(0,0) + \tfrac{1}{3}|g_t|^{\frac{4}{3}-\Delta_3} Q_v(0,0) + \dots, \tag{10}$$

where $Q_u(0,0)$ and $Q_v(0,0)$ denote respectively the partial derivatives of $Q(u, v)$ with respect to u and v evaluated at $u = v = 0$.

The following conclusions are now immediate:

(i) M_0 and P_0 both vanish at $K_{3,c}$ with the same exponent $\beta = 4/3 - \Delta = 1/12$.

(ii)
$$\lim_{K_3 \to K_{3,c}+} (3P_0/M_0) = a. \tag{11}$$

(iii) Unless $Q_v(0,0) \equiv 0$, the exponent Δ_3 and hence y_3 will be evident in the corrections to the leading power law behaviour, although in principle there could be other terms arising from corrections to (2). Moreover, since $y_t > y_3 > 0$, the exponent, ω, of the correction term must be in the interval $4/3 > \omega > 1/3$.

While neither M_0 nor P_0 has yet been evaluated exactly, Baxter *et al* (1975) have conjectured very plausible functional forms valid for the whole range $K_{3,c} \leq K < \infty$. Expanding these results we find

$$M_0 = 2^{1/3} \epsilon^{1/12} [1 + \tfrac{1}{24}\epsilon - 2^{-4/3} \epsilon^{2/3} + O(\epsilon^{4/3})] \tag{12}$$

$$P_0 = \tfrac{1}{3} 2^{11/6} \epsilon^{1/12} [1 + \tfrac{7}{8}\epsilon + O(\epsilon^2)], \tag{13}$$

where

$$\epsilon = 1 - (\sinh 2K_3)^{-1} = O(K_3 - K_{3,c}). \tag{14}$$

Equations (12) and (13) immediately confirm the first conclusion above, while (11) implies that

$$a = 2\sqrt{2} = 2{\cdot}828 \dots. \tag{15}$$

The corresponding estimates of den Nijs *et al* (1976) are 2·815 from a four-cell cluster approximation and 2·737 from a six-cell approximation, which agree rather well with (15). Imbro and Hemmer (1976) do not report a value for a.

If we attribute the linear terms inside the parentheses in (12) to quadratic corrections to the scaling fields (5), comparison of (12) with (9) yields

$$\Delta_3 = 7/12, \qquad \text{i.e.} \quad y_3 = 7/8. \tag{16}$$

The problem now is the absence of any similar correction term through $O(\epsilon^2)$ in P_0. The only feasible way out of this appears to be to modify (2) by assuming

$$-\beta f_s = |g_t|^{4/3} Q(u, v) + |g_t|^2 Q_1(u, v) \tag{17}$$

with $v = g_3/|g_t|^{7/12}$. If we now *demand* that

$$Q_v(0,0) + aQ_{1u}(0,0) = 0, \tag{18}$$

the correction term of $O(\epsilon^{3/4})$ in P_0 vanishes. The corresponding term in M_0 has amplitude

$$bQ_v(0,0) + Q_{1u}(0,0) = -\tfrac{1}{2}, \tag{19}$$

on comparing with (12).

Unfortunately it does not seem possible to independently determine either $Q_v(0, 0)$ or $Q_{1u}(0, 0)$ and so determine b. One possibility is to attempt to match higher-order correction terms in the expansions (12) and (13) to (18). However, it seems impossible to unravel from these terms the contributions arising directly from (18) and those arising from the non-linear corrections in (5). In addition, if one expands the specific heat result of Baxter and Wu (1973, 1974), (18) only encompasses the specific heat correction terms if it is again extended to include a term of order $|g_t|^{8/3}$.

This final conclusion suggests that the asymptotic behaviour of $-\beta f_s$ in the vicinity of $K_3 = K_{3,c}$, $K_1 = K_2 = 0$ is actually of the form

$$-\beta f_s \simeq |g_t|^{4/3} \hat{Q}(u, v, w), \tag{20}$$

with

$$u = g_h |g_t|^{-5/12}, \qquad v = g_3 |g_t|^{-7/12}, \qquad w = g_i |g_t|^{2/3}, \tag{21}$$

where g_i is a non-vanishing irrelevant field with exponent $y_i = -1$. Since different values of such a field do not affect the critical properties, this suggests that the critical surface associated with the Baxter–Wu type fixed points does not consist of a sole point as found by den Nijs *et al* (1976) and Imbro and Hemmer (1976). Clearly improved renormalization group calculations are required to explore this aspect in more detail.

The exact nature of the critical surface is however irrelevant to our basic conclusion, that the corrections to the asymptotic power law behaviour of $M_0(K_3)$ and $P_0(K_3)$ appear to be only consistent with a conventional scaling *ansatz* if the Baxter–Wu critical point is associated with a fixed point with *three* relevant eigenvalues,

$$y_t = 3/2, \qquad y_h = 15/8, \qquad y_3 = 7/8. \tag{22}$$

This value for y_3 is rather larger than the estimate of $y_3 \simeq 0 \cdot 3$ made by den Nijs *et al* (1976) and rather less than that found by Imbro and Hemmer (1976) ($y_3 \simeq 1 \cdot 2$). Presumably this discrepancy reflects the basic inaccuracy of these calculations and should be decreased by more effective approximations.

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